

THE STRUCTURE OF MOTIVIC HOMOTOPY GROUPS

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ABSTRACT. We study the stable motivic homotopy groups $\pi_{s,w}$ of the 2-completion of the motivic sphere spectrum over \mathbb{C} . When arranged in the (s, w) -plane, these groups break into four different regions: a vanishing region, an η -local region that is entirely known, a τ -local region that is identical to classical stable homotopy groups, and a region that is not well-understood.

1. INTRODUCTION

This article is concerned with the motivic stable homotopy groups over \mathbb{C} . More specifically, we consider the motivic stable homotopy groups $\pi_{s,w}$ of the 2-completion of the motivic sphere spectrum, where s is the stem and w is the motivic weight. Motivic completion behaves somewhat differently than classical completion. In particular, the homotopy groups of the completed motivic sphere are not necessarily the same as the completions of the integral motivic homotopy groups. In fact, one needs to be careful about completion with respect to the Hopf map η as well [4].

The bigraded nature of the motivic homotopy groups leads to a natural arrangement in the (s, w) -plane. Our main result is that this plane breaks into four distinct regions. These regions are indicated in Figure 1. For clarity, the figure is not to scale.

Beware that the grading in Figure 1 does not follow the usual Adams style. The vertical axis plots the motivic weight, not a spectral sequence filtration. On the other hand, the Adams and Adams-Novikov charts in [6] [7] and elsewhere suppress the motivic weight and show spectral sequence filtrations on the vertical axis. The reader should be careful not to confuse the different conventions.

We will now describe the qualitative behavior of motivic homotopy groups in each region of Figure 1.

The vanishing region. The group $\pi_{s,w}$ is zero when $s < 0$ or $w > s$. This is closely related to Morel's connectivity theorem for motivic homotopy theory [10]. We will give a different proof in Theorem 4.2 that is consistent with the methods of this article. The boundaries of the vanishing region are sharp in the sense that there are infinitely families of non-zero elements that lie on the line $s = 0$ and on the line $w = s$.

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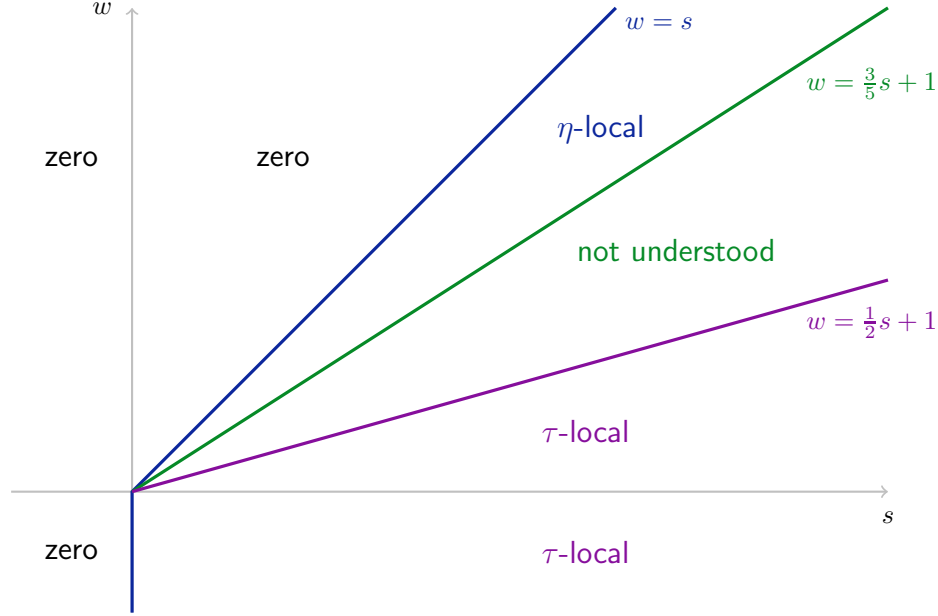


FIGURE 1. Homotopy groups $\pi_{s,w}$ of the 2-completed motivic sphere over \mathbb{C} (not to scale).

The τ -local region. Levine [9, Theorem 6.7] showed that the realization functor from motivic homotopy theory over \mathbb{C} to classical homotopy theory induces an isomorphism

$$\pi_{s,0} \xrightarrow{\cong} \pi_s$$

from motivic homotopy groups in weight zero to classical homotopy groups (even before 2-completion). This means that the classical groups π_s lie on the positive s -axis in Figure 1.

Recall that τ is an element of $\pi_{0,-1}$ that induces multiplication by τ in motivic cohomology. The realization functor from motivic homotopy theory over \mathbb{C} to classical homotopy theory is well-understood computationally. According to [8, Proposition 3.0.2], applying this functor has the effect of inverting τ . We use the motivic Adams-Novikov spectral sequence to show that the maps

$$\pi_{s,w} \rightarrow \pi_{s,w-1}$$

given by multiplication by τ are isomorphisms when $w \leq \frac{1}{2}s + 1$ (and when $w \leq 0$ if $s = 0$). Therefore, each group $\pi_{s,w}$ in this region is isomorphic to its τ -localization, which in turn is isomorphic to the classical (2-completed) homotopy group π_s .

The slope of the boundary of the τ -local region is sharp in the sense that there is an infinite family of non-zero elements that lie on a line of slope $\frac{1}{2}$ and are annihilated by τ .

The η -local region. The motivic Hopf map η in $\pi_{1,1}$ is not nilpotent. The values of the localized groups $\pi_{*,*}[\eta^{-1}]$ were conjectured in [3] and proved in [1]. See Proposition 2.4 for an explicit description of the calculation. We use results from

[1] to show that every element of $\pi_{s,w}$ is η -local when $w > \frac{3}{5}s + 1$ and $w \leq s$. This means that the maps

$$\pi_{s,w} \rightarrow \pi_{s+1,w+1}$$

given by multiplication by η are isomorphisms in this region. As a result, every group in the η -local region is explicitly known.

The upper boundary of the η -local region is sharp, in the sense that there is an η -local family of non-zero elements that lie on the upper boundary line. The slope of the lower boundary is sharp in the sense that there is an infinite family of non-zero elements that lie on a line of slope $\frac{3}{5}$ and are annihilated by η .

The not understood region. The final region consists of groups $\pi_{s,w}$ for which $w \leq \frac{3}{5}s + 1$ and $w > \frac{1}{2}s + 1$. All of the exotic motivic phenomena fall within this relatively small slice whose boundary lines meet at an angle of less than 5 degrees. Further study of motivic stable homotopy groups over \mathbb{C} has the aim of better understanding this region.

2. THE CLASSICAL AND MOTIVIC ADAMS-NOVIKOV SPECTRAL SEQUENCES

We will work in the same framework as [8], that is, over \mathbb{C} and at the prime 2. Our results extend to any other algebraically closed field of characteristic zero.

Let $E_r(S^0; BP)$ be the E_r page of the classical Adams-Novikov spectral sequence, and let $E_r(S^{0,0}; BPL)$ be the E_r page of the motivic Adams-Novikov spectral sequence for the 2-completed motivic sphere. In the motivic context, we use degrees of the form (s, f, w) , where s is the stem, f is the Adams-Novikov filtration, and w is the weight. In the classical context, we use degrees of the form (s, f) , where s is the stem and f is the Adams-Novikov filtration. An element in degree (s, f, w) occurs at location (s, f) in a traditional Adams-Novikov chart.

We next describe the relationship between the motivic E_2 page and the classical E_2 page. First define an intermediate object

$$\overline{E}_2^{s,f,w}(S^{0,0}; BPL) = \begin{cases} E_2^{s,f}(S^0; BP) & \text{if } w = \frac{s+f}{2} \\ 0 & \text{if } w \neq \frac{s+f}{2}. \end{cases}$$

Proposition 2.1 ([5, Theorem 8 and Section 4], [8, Theorem 6.1.4]). *There is a tri-graded isomorphism*

$$E_2(S^{0,0}; BPL) \cong \overline{E}_2(S^{0,0}; BPL) \otimes_{\mathbb{Z}} \mathbb{Z}[\tau]$$

where τ has degree $(0, 0, -1)$.

In other words, the motivic Adams-Novikov E_2 page is completely determined by the classical Adams-Novikov E_2 page. We rephrase Proposition 2.1 in an even more explicit form.

Corollary 2.2. *As a graded abelian group, the motivic Adams-Novikov E_2 page is given by*

$$E_2^{s,f,w}(S^{0,0}; BPL) = \begin{cases} E_2^{s,f}(S^0; BP) & \text{if } w \leq \frac{s+f}{2} \\ 0 & \text{if } w > \frac{s+f}{2}. \end{cases}$$

Let α_1 be the classical element of $E_2^{1,1}(S^0; BP)$ that represents the Hopf map η in π_1 . We abuse notation and write α_1 for its motivic analogue in $E_2^{1,1,1}(S^{0,0}; BPL)$ that represents the motivic Hopf map η in $\pi_{1,1}$. Define $\alpha_1^{-1}E_2^{s,f}(S^0; BP)$ to be

$$\text{colim} \left(E_2^{s,f}(S^0; BP) \xrightarrow{\cdot \alpha_1} E_2^{s+1,f+1}(S^0; BP) \xrightarrow{\cdot \alpha_1} \dots \right).$$

Similarly, define $\alpha_1^{-1}E_2^{s,f,w}(S^{0,0}; BPL)$ to be

$$\operatorname{colim} \left(E_2^{s,f,w}(S^{0,0}; BPL) \xrightarrow{\cdot\alpha_1} E_2^{s+1,f+1,w+1}(S^{0,0}; BPL) \xrightarrow{\cdot\alpha_1} \dots \right).$$

The groups $\alpha_1^{-1}E_2^{s,f}(S^0; BP)$ assemble into a localized Adams-Novikov spectral sequence which, by a vanishing line argument, converges to the classical homotopy groups $\pi_*S^0[\eta^{-1}]$ of the η -localized classical sphere. These localized groups are zero since η^4 is zero classically. Similarly, the groups $\alpha_1^{-1}E_2^{s,f,w}(S^{0,0}; BPL)$ assemble into a localized motivic Adams-Novikov spectral sequence that converges to $\pi_{*,*}S^{0,0}[\eta^{-1}]$, which will be described below in Proposition 2.4.

Proposition 2.3. *The localization map*

$$E_2^{s,f,w}(S^{0,0}; BPL) \longrightarrow \alpha_1^{-1}E_2^{s,f,w}(S^{0,0}; BPL)$$

is an isomorphism for all weights w and for $s < 5f - 10$.

Proof. By [1, Proposition 5.1], the localization map

$$E_2^{s,f}(S^0; BP) \longrightarrow \alpha_1^{-1}E_2^{s,f}(S^0; BP)$$

is an isomorphism in the range $s < 5f - 10$. The motivic analogue follows from the identifications of Corollary 2.2. \square

For reference, we give a complete description of the α_1 -localized motivic Adams-Novikov spectral sequence. We are mostly interested in the E_∞ page, which gives the homotopy of the η -local motivic sphere.

Proposition 2.4 ([1, Corollary 7.1 and Theorem 7.2]).

- (1) *The E_2 page of the localized motivic Adams-Novikov spectral sequence is*

$$\alpha_1^{-1}E_2^{*,*,*}(S^{0,0}; BPL) \cong \mathbb{F}_2[\tau, \alpha_1^{\pm 1}, \alpha_3, \alpha_4] / \alpha_4^2,$$

where τ has degree $(0, 0, -1)$; α_1 has degree $(1, 1, 1)$; α_3 has degree $(5, 1, 3)$; and α_4 has degree $(7, 1, 4)$.

- (2) *All differentials are deduced via the Leibniz rule from $d_3(\alpha_3) = \tau\alpha_1^4$, and*

$$\alpha_1^{-1}E_\infty^{s,f,w}(S^{0,0}; BPL) \cong \mathbb{F}_2[\alpha_1^{\pm 1}, \alpha_3^2, \alpha_4] / \alpha_4^2.$$

- (3) *The homotopy of the η -localized motivic sphere is given by*

$$\pi_{*,*}S^{0,0}[\eta^{-1}] \cong \mathbb{F}_2[\eta^{\pm 1}, \sigma, \mu_9] / \sigma^2,$$

where η in degree $(1, 1)$ is detected by α_1 ; σ in degree $(7, 4)$ is detected by α_4 ; and μ_9 in degree $(9, 5)$ is detected by $\alpha_1^{-1}\alpha_3^2$.

3. THE COFIBER OF τ

Recall that τ is a map $S^{0,-1} \longrightarrow S^{0,0}$ that induces multiplication by τ in motivic cohomology. Let $C\tau$ be the cofiber of this map. We have a cofiber sequence

$$(1) \quad S^{0,-1} \xrightarrow{\tau} S^{0,0} \longrightarrow C\tau \longrightarrow S^{1,-1}.$$

We will use the spectrum $C\tau$ to obtain information about the homotopy groups of the motivic sphere.

Using the relationship between the motivic and classical Adams-Novikov spectral sequences, we get the following remarkable description of the homotopy groups of $C\tau$.

Proposition 3.1 ([8, Proposition 6.2.5]). *The homotopy of $C\tau$ is given by*

$$\pi_{s,w}(C\tau) \cong E_2^{s,2w-s,w}(S^{0,0}; BPL) \cong E_2^{s,2w-s}(S^0; BP).$$

Proof. For reference, we reproduce the short proof from [8]. The defining cofiber sequence (1) induces a long exact sequence

$$\dots \longrightarrow E_2(S^{0,-1}; BPL) \xrightarrow{\tau} E_2(S^{0,0}; BPL) \longrightarrow E_2(C\tau; BPL) \longrightarrow \dots$$

on motivic Adams-Novikov E_2 pages. Multiplication by τ is injective by Proposition 2.1, so $E_2(C\tau; BPL)$ is isomorphic to $\overline{E}_2(S^{0,0}; BPL)$. Therefore, $E_2(C\tau; BPL)$ is concentrated in degrees of the form $(s, f, \frac{s+f}{2})$. For degree reasons, there can be no differentials, so the spectral sequence collapses to $E_\infty(C\tau; BPL) \cong \overline{E}_2(S^{0,0}; BPL)$. Moreover, for similar degree reasons, no hidden extensions are possible. \square

Proposition 3.1 says essentially that the motivic homotopy groups of $C\tau$ are isomorphic to the classical Adams-Novikov E_2 page as bigraded objects.

Corollary 3.2. *The group $\pi_{s,w}(C\tau)$ is zero when $w \leq \frac{1}{2}s$, except that $\pi_{0,0}(C\tau) = \mathbb{Z}_2$.*

Proof. By Proposition 3.1, we have an isomorphism $\pi_{s,w}(C\tau) \cong E_2^{s,2w-s}(S^0; BP)$. When $2w - s \leq 0$, the group $E_2^{s,2w-s}(S^0; BP)$ is zero, except that $E_2^{0,0}(S^0; BP)$ equals \mathbb{Z}_2 . \square

4. THE VANISHING AND LOCAL REGIONS

The goal of this section is to make precise the results on each region of Figure 1 that were described informally in Section 1.

4.1. The vanishing region. We first describe the region to the left of the line $s = 0$ or above the line $w = s$.

Theorem 4.2. *The group $\pi_{s,w}$ is zero if $w > s$ or $s < 0$.*

Proof. We begin with the motivic May spectral sequence; see [8, Chapter 2] for more details. The E_1 page is a polynomial algebra over $\mathbb{F}_2[\tau]$ with generators h_{ij} for all $i > 0$ and $j \geq 0$. The element h_{i0} lies in stem $2^i - 2$ and weight $2^{i-1} - 1$, while the element h_{ij} lies in stem $2^j(2^i - 1) - 1$ and weight $2^{j-1}(2^i - 1)$ for $j > 0$. In every case, the weight is always less than or equal to the stem. Therefore, the entire May E_1 page vanishes in degrees where the weight is greater than the stem.

The target of the motivic May spectral sequence is the E_2 page of the motivic Adams spectral sequence. Thus the Adams E_2 page also vanishes in degrees where the weight is greater than the stem.

The target of the motivic Adams spectral sequence are the homotopy groups of the 2-completed motivic sphere. Finally, these homotopy groups vanish in degrees where the weight is greater than the stem.

A similar argument shows that the homotopy groups vanish in negative stems. \square

Remark 4.3. Both of the boundary lines in Theorem 4.2 are sharp in the following sense. The non-zero elements η^k form an infinite family on the line $w = s$, and the non-zero elements τ^k form an infinite family on the line $s = 0$.

4.4. The τ -local region. We next describe the region below the line $w = \frac{1}{2}s + 1$ and to the right of the line $s = 0$.

Theorem 4.5. *If $w \leq \frac{1}{2}s + 1$ and $s > 0$, or if $w \leq 0$ and $s = 0$, then the map*

$$\pi_{s,w} \longrightarrow \pi_{s,w-1}$$

given by multiplication by τ is an isomorphism. Moreover, in this range $\pi_{s,w}$ is isomorphic to the classical group π_s .

Proof. For $s = 0$, we know that $\pi_{0,*}$ is equal to $\mathbb{Z}_2[\tau]$, where τ has degree $(0, -1)$. In Figure 1, these elements are represented by countably many copies of \mathbb{Z}_2 on the negative w axis. Since the classical group π_0 is isomorphic to \mathbb{Z}_2 , this settles the case $s = 0$.

Now we may assume that $s > 0$. Consider the long exact sequence

$$\cdots \longrightarrow \pi_{s+1,w-1}(C\tau) \longrightarrow \pi_{s,w} \xrightarrow{\tau} \pi_{s,w-1} \longrightarrow \pi_{s,w-1}(C\tau) \longrightarrow \cdots$$

of homotopy groups obtained from the cofiber sequence (1). By the vanishing result of Corollary 3.2, the long exact sequence becomes

$$(2) \quad \cdots \longrightarrow 0 \longrightarrow \pi_{s,w} \xrightarrow{\tau} \pi_{s,w-1} \longrightarrow 0 \longrightarrow \cdots$$

when $w \leq \frac{1}{2}s + 1$. Therefore multiplication by τ is an isomorphism in this range.

The final claim follows from the fact that realization from motivic homotopy theory to classical homotopy theory induces τ -localization on homotopy groups [8, Proposition 3.0.2]. \square

Example 4.6. Consider the elements $P^k h_1^4$ of the motivic Adams spectral sequence in degrees $(4, 4, 4) + k(8, 4, 4)$. These elements detect homotopy classes that are annihilated by τ . This family lies on the line $w = \frac{1}{2}s + 2$. Therefore, the slope of the line in Theorem 4.5 cannot be improved.

4.7. The η -local region. Now we consider the region above the line $w = \frac{3}{5}s + 1$ and below the line $w = s$.

Theorem 4.8. *If $w > \frac{3}{5}s + 1$, then the map*

$$\pi_{s,w} \longrightarrow \pi_{s+1,w+1}$$

given by multiplication by η is an isomorphism.

Proof. By Proposition 2.1, any non-trivial element in the motivic Adams-Novikov spectral sequence satisfies the inequality $s + f - 2w \geq 0$. Under this condition, our constraint $w > \frac{3}{5}s + 1$ implies that $s < 5f - 10$. By Proposition 2.3, in this region the motivic Adams-Novikov spectral sequence agrees with the α_1 -localized motivic Adams-Novikov spectral sequence, and thus the survivors are η -local as claimed. \square

Remark 4.9. In the motivic Adams spectral sequence, every element above the line $f = \frac{1}{2}s + 2$ is h_1 -local [2]. Using that the motivic Adams spectral sequence vanishes when $s + f - 2w < 0$ [8, Remark 2.1.13], we can conclude that $\pi_{s,w}$ is η -local if $w \geq \frac{3}{4}s + 1$. This result is strictly weaker than the result of Theorem 4.8.

Remark 4.10. Proposition 2.4 explicitly describes the homotopy groups of the η -local sphere. In particular, every homotopy group in the η -local region is completely understood.

Example 4.11. Recent work of Michael Andrews and others shows that the elements $h_2^3 g^k$ of the motivic Adams spectral sequence in degrees $(9, 3, 6) + k(20, 4, 12)$ are non-trivial permanent cycles for all $k \geq 0$. The associated homotopy classes form a “ w_1 -periodic” family that lie on the line $w = \frac{3}{5}s + \frac{3}{5}$, and they are all annihilated by η . This shows that the slope of the bottom line in Theorem 4.8 cannot be improved.

On the other hand, the elements η^k lie on the line $w = s$, which shows that the top line in Theorem 4.8 is sharp.

Remark 4.12. Theorem 4.8 shows that every element that lies above the line $w = \frac{3}{5}s + 1$ and below the line $w = s$ is η -local. However, there are η -local elements that lie outside this region. For example, consider the elements $P^k h_1$ of the motivic Adams spectral sequence in degree $(1, 1, 1) + k(8, 4, 4)$. These elements detect homotopy classes that are η -local and lie below the line $w = \frac{3}{5}s + 1$.

4.13. The not understood region. Finally, we come to the region above the line $w = \frac{1}{2}s$ and below the line $w = \frac{3}{5}s + 1$. All of the exotic behavior of motivic homotopy groups lies in this region.

For example, there is an exotic non-nilpotent element in $\pi_{32,18}$. This element and all of its powers lie in the not understood region.

Motivic v_n -self maps of period k have bidegree $k(2^{n+1} - 2, 2^n - 1)$. Therefore, all v_n -periodic families in $\pi_{*,*}$ lie on lines of slope $1/2$. These families are parallel to the bottom edge of the not understood region.

Recent work of Andrews and others explores “ w_n -periodicity” in motivic homotopy groups. Motivic w_n -self maps of period k have bidegree $k(2^{n+2} - 3, 2^{n+1} - 1)$. For example, w_0 has degree $(1, 1)$ and in fact is the same as multiplication by η . Next, w_1 has degree $(5, 3)$. Thus, w_1 -periodic families, such as the one in Example 4.11, lie on lines of slope $\frac{3}{5}$. Since w_2 has degree $(13, 7)$, we speculate that there is a line of slope $\frac{7}{13}$ such that all elements above this line are w_0 -periodic (i.e., η -local) or w_1 -periodic.

As n increases, w_n -periodic families lie on lines whose slopes approach $\frac{1}{2}$ from above. It is conceivable that w_n -periodicity will lead to a better understanding of the large-scale structure of the not understood region.

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